Contact resolutions of projectivised nilpotent orbit closures

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Abstract

The projectivised nilpotent orbit closure $\mathbb{P}(\overline{\mathcal{O}})$ carries a natural contact structure on its smooth part, which is induced by a line bundle L on $\mathbb{P}(\overline{\mathcal{O}})$. A resolution $\pi : X \to \mathbb{P}(\overline{\mathcal{O}})$ is called *contact* if $\pi^* L$ is a contact line bundle on X. It turns out that contact resolutions, crepant resolutions and minimal models of $\mathbb{P}(\overline{\mathcal{O}})$ are all the same. In this note, we determine when $\mathbb{P}(\overline{\mathcal{O}})$ admits a contact resolution, and in the case of existence, we study the birational geometry among different contact resolutions.

1 Introduction

Recall that a nilpotent orbit $\mathcal{O}$ in a semi-simple complex Lie algebra $\mathfrak{g}$ enjoys the following properties:

(i) it is $\mathbb{C}^*$-invariant, where $\mathbb{C}^*$ acts on $\mathfrak{g}$ by linear scalars;

(ii) it carries the Kirillov-Kostant-Souriau symplectic 2-form ω;

(iii) $\lambda^*\omega = \lambda\omega$ for any $\lambda \in \mathbb{C}^*$.

One deduces from (iii) that this symplectic structure on $\mathcal{O}$ gives a contact structure on the projectivisation $\mathbb{P}(\mathcal{O})$, which is induced by the line bundle $L := \mathcal{O}_{\mathbb{P}(\mathfrak{g})}(1)|_{\mathbb{P}(\overline{\mathcal{O}})}$. When $\mathfrak{g}$ is simple, the variety $\mathbb{P}(\mathcal{O}) \subset \mathbb{P}(\mathfrak{g})$ is closed if and only if $\mathcal{O}$ is the minimal nilpotent orbit $\mathcal{O}_{min}$ (see for example Prop. 2.6 [Be]). In this case, $\mathbb{P}(\mathcal{O}_{min})$ is a Fano contact manifold. It is generally believed that these are the only examples of such varieties ([Be], [Le1]). A positive answer to this would imply that every compact quaternion-Kähler manifold with positive scalar curvature is homothetic to a Wolf space (Theorem 3.2 [LeSa]).

If we take the closure $\overline{\mathbb{P}(\mathcal{O})} = \mathbb{P}(\overline{\mathcal{O}})$, then it is in general singular. We say that a resolution $\pi : X \to \mathbb{P}(\overline{\mathcal{O}})$ is *contact* if $\pi^* L$ is a contact line bundle on X. It follows that X is a projective contact manifold. Such varieties have drawn much attention recently(see for example [Pe] and the references therein).

The first aim of this note is to find all contact resolutions that $\mathbb{P}(\overline{\mathcal{O}})$ can have. More precisely we prove that (Theorem 4.5) if the normalization $\mathbb{P}(\widetilde{\mathcal{O}})$ of $\mathbb{P}(\overline{\mathcal{O}})$ is not smooth, then the resolution X is isomorphic to $\mathbb{P}(T^*(G/P))$ for some parabolic sub-group P in the adjoint group G of $\mathfrak{g}$ and π is the natural resolution. The proof relies on the main result in [KPSW] and that in [Fu1]. A classification (Corollary 4.6) of $\mathcal{O}$ such that $\mathbb{P}(\overline{\mathcal{O}})$ admits a contact resolution can be derived immediately, with the help of [Be].

Once we have settled the problem of existence of a contact resolution, we turn to study the birational geometry among different contact resolutions in the last section, where (Theorem 5.2) the chamber structure of the movable cone of a contact resolution is given, based on the main result in [Na]. This gives another way to prove the aforesaid result under the condition that $\overline{\mathcal{O}}$ admits a symplectic resolution, since minimal models, contact resolutions and crepant resolutions of $\mathbb{P}(\overline{\mathcal{O}})$ are the same objects (Proposition 3.3).

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2 Singularities in $\mathbb{P}(\widetilde{\mathcal{O}})$

Let $\mathfrak{g}$ be a simple complex Lie algebra and $\mathcal{O}$ a nilpotent orbit in $\mathfrak{g}$. The normalization of the closure $\overline{\mathcal{O}}$ will be denoted by $\widetilde{\mathcal{O}}$. The scalar $\mathbb{C}^*$-action on $\overline{\mathcal{O}}$ lifts to $\widetilde{\mathcal{O}}$. There is only one $\mathbb{C}^*$-fixed point on $\widetilde{\mathcal{O}}$, say o. We denote by $\mathbb{P}(\widetilde{\mathcal{O}})$ the geometric quotient of $\widetilde{\mathcal{O}} \setminus \{o\}$ by the $\mathbb{C}^*$-action. Similarly we denote by $\mathbb{P}(\overline{\mathcal{O}})$ the geometric quotient $\overline{\mathcal{O}} \setminus \{0\}//\mathbb{C}^*$. Note that $\mathbb{P}(\widetilde{\mathcal{O}})$ is nothing but the normalization of $\mathbb{P}(\overline{\mathcal{O}})$.

Recall that a *contact structure* on a smooth variety X is a corank 1 subbundle $F \subset TX$ such that the O'Neil tensor $F \times F \to L := TX/F$ is everywhere non-degenerate. In this case, L is called a contact line bundle on X and we have $K_X \simeq L^{-(n+1)}$, where $n = (dim X - 1)/2$. If we regard the natural map $TX \to L$ as a section $\theta \in H^0(X, \Omega^1_X(L))$ (called a *contact form*), then the non-degenerateness is equivalent to the condition that $\theta \wedge (d\theta)^n$ is nowhere vanishing when considered locally as an element in $H^0(X, \Omega_X^{2n+1}(L^{n+1})) = H^0(X, \mathcal{O}_X)$.

For a point $v \in \mathcal{O}$, the tangent space $T_v\mathcal{O}$ is naturally isomorphic to $[v, \mathfrak{g}] = Img\ ad_v$. The map $[v, x] \mapsto \kappa(v, x)$ defines a one-form θ' on $\mathcal{O}$, where κ is the Killing form of $\mathfrak{g}$. Then $\omega := d\theta'$ is the Kirillov-Kostant-Souriau symplectic form on $\mathcal{O}$. Notice that $\lambda^*\theta' = \lambda\theta'$ for every $\lambda \in \mathbb{C}^*$, so it defines an element $\theta \in H^0(\mathbb{P}(\mathcal{O}), \Omega^1_{\mathbb{P}(\mathcal{O})}(L))$, where L is the pull-back of $\mathcal{O}_{\mathbb{P}(\mathfrak{g})}(1)$ to $\mathbb{P}(\mathcal{O})$. This is in fact a contact form, i. e. $\theta \wedge (d\theta)^{\wedge n}$ is everywhere non-zero, where $n = (dim\mathcal{O} - 2)/2$. Since the codimension of the complement of $\mathbb{P}(\mathcal{O})$ in $\mathbb{P}(\widetilde{\mathcal{O}})$ is at least 2, θ extends to a contact form on the smooth part of $\mathbb{P}(\widetilde{\mathcal{O}})$.

Remark 2.1. Let G be the adjoint group of $\mathfrak{g}$. Then the contact structure on $\mathbb{P}(\mathcal{O})$ is G-invariant, which is precisely the contact structure on $\mathbb{P}(\mathcal{O})$ when $\mathbb{P}(\mathcal{O})$ is viewed as a twistor space of a quaternion-Kähler manifold ([Sw]).

Proposition 2.1. *The projective variety $\mathbb{P}(\widetilde{\mathcal{O}})$ is projectively normal with only rational Gorenstein singularities.*

Proof. By abusing the notations, we denote also by L the pull-back of L to the normalization $\mathbb{P}(\widetilde{\mathcal{O}})$, which is a line bundle. Note that the complement of $\mathbb{P}(\mathcal{O})$ in $\mathbb{P}(\widetilde{\mathcal{O}})$ has codimension at least 2, so $K_{\mathbb{P}(\widetilde{\mathcal{O}})} = L^{-n-1}$ is locally free, which implies that $\mathbb{P}(\widetilde{\mathcal{O}})$ is Gorenstein. Notice that $\widetilde{\mathcal{O}} \setminus \{o\}$ has rational singularities by results of Panyushev and Hinich (see [Pa]), so its quotient by the $\mathbb{C}^*$ action $\mathbb{P}(\widetilde{\mathcal{O}})$ has only rational Gorenstein singularities. $\square$

The following proposition is easily deduced from Proposition 5.2 in [Be], which plays an important role to our classification result (Corollary 4.6).

Proposition 2.2. *Let $\mathfrak{g}$ be a simple Lie algebra and $\mathcal{O} \subset \mathfrak{g}$ a non-zero nilpotent orbit. Then $\mathbb{P}(\widetilde{\mathcal{O}})$ is smooth if and only if either $\mathcal{O}$ is the minimal nilpotent orbit or $\mathfrak{g}$ is of type G_2 and $\mathcal{O}$ is the nilpotent orbit of dimension 8.*

Singularities in $\mathbb{P}(\widetilde{\mathcal{O}})$ are examples of the so-called *contact singularities* in [CF]. Projectivised nilpotent orbits have already been studied, for example, in [Be] (for relation with Fano contact manifolds), [Ko] (for relation with harmonic maps) and [Sw] (from the twistor aspect). Their closures have also been studied, for example in [Po] (for the self-duality), which give examples of non-smooth, self-dual projective varieties.

3 Minimal models

For a proper morphism between normal varieties $f : X \to W$, we denote by $N_1(f)$ the vector space (over $\mathbb{R}$) generated by reduced irreducible curves contained in fibers of f modulo numerical equivalence. Let $N^1(f)$ be the group $Pic(X) \otimes \mathbb{R}$ modulo numerical equivalence (w. r. t. $N_1(f)$), then we have a perfect pairing $N_1(f) \times N^1(f) \to \mathbb{R}$.

If f is a resolution, then X is called a *minimal model* of W if K_X is f-nef.

Proposition 3.1. *Let W be a projective normal variety with only canonical singularities and $f : X \to W$ a resolution. Then f is crepant if and only if X is a minimal model of W.*

Proof. If f is crepant, then $K_X = f^*K_W$, which gives $K_X \cdot [C] = 0$ for every f-exceptional curve C, so X is a minimal model of W.

Suppose K_X is f-nef, then so is $K_X - f^*K_W = \sum_i a_i E_i$, where E_i are exceptional divisors of f. By the negativity lemma (see Lemma 13-1-4 [Ma]), $a_i \leq 0$ for all i. On the other hand, W has only canonical singularities, so $a_i \geq 0$, which gives $a_i = 0$ for all i, thus f is crepant. $\square$

Corollary 3.2. *Let W be a projective normal variety with only terminal singularities and $f : X \to W$ a resolution. Then the following statements are equivalent:*

(i) f is crepant;
(ii) X is a minimal model of W;
(iii) f is small, i.e. $codim(Exc(f)) \geq 2$.

By the previous section, there is a contact structure on $\mathbb{P}(\mathcal{O})$, induced by the line bundle L on $\mathbb{P}(\overline{\mathcal{O}})$. A *contact resolution* of $\mathbb{P}(\widetilde{\mathcal{O}})$ is a resolution $\pi : X \to \mathbb{P}(\widetilde{\mathcal{O}})$ such that π^*L is a contact line bundle on X.

Proposition 3.3. *Let $\pi : X \to \mathbb{P}(\widetilde{\mathcal{O}})$ be a resolution, then the following statements are equivalent:*
(i) π is crepant;
(ii) K_X is π-nef;
(iii) π is a contact resolution.

Proof. The equivalence between (i) and (ii) follows from Prop. 2.1 and Prop. 3.1. The implication (iii) to (i) is clear from the definitions. Now suppose that π is crepant, then $K_X \simeq \pi^*(L^{-(n+1)}) \simeq (\pi^* L)^{-(n+1)}$. Let $\hat{X}$ be the fiber product $X \times_{\mathbb{P}(\widetilde{\mathcal{O}})} (\widetilde{\mathcal{O}} \setminus \{o\})$ and $h : \hat{X} \to \widetilde{\mathcal{O}} \setminus \{o\}$ the natural projection. Then h is a resolution of singularities and $h^*\omega$ extends to a 2-form $\tilde{\omega}$ on $\hat{X}$ since $\widetilde{\mathcal{O}} \setminus \{o\}$ has only symplectic singularities, where ω is the symplectic form on the smooth part of $\widetilde{\mathcal{O}}$. $\hat{X}$ inherits a $\mathbb{C}^*$-action from that on $\widetilde{\mathcal{O}}$. Contracting $\tilde{\omega}$ with the vector field generating the $\mathbb{C}^*$-action, one obtains an element $\tilde{\theta} \in H^0(X, \Omega_X \otimes \pi^* L)$. Now it is clear that $\tilde{\theta}$ gives the contact form on X extending θ. □

4 Contact resolutions

Let $f : Z \to \mathbb{P}(\overline{\mathcal{O}})$ be a resolution and let $\hat{Z}$ be the fiber product $Z \times_{\mathbb{P}(\overline{\mathcal{O}})} W_0$ and $\tilde{f} : \hat{Z} \to W_0$ the natural projection, where $W_0 = \overline{\mathcal{O}} \setminus \{0\}$. Recall that L is the restriction of $\mathcal{O}_{\mathbb{P}(\mathfrak{g})}(1)$ to $\mathbb{P}(\overline{\mathcal{O}})$.

Lemma 4.1. *$\hat{Z}$ is isomorphic to the complement of the zero section in the total space of the line bundle $(f^*L)^*$ and $\tilde{f}$ is a resolution of singularities.*

Proof. This follows from that W_0 is naturally isomorphic to the complement of the zero section in L^* and the fiber product $Z \times_{\mathbb{P}(\overline{\mathcal{O}})} L^*$ is isomorphic to $f^*(L^*) \simeq (f^*L)^*$. □

Proposition 4.2. *The map f is a contact resolution if and only if $\tilde{f}$ is a symplectic resolution.*

Proof. Let ω be the Kostant-Kirillov-Souriau symplectic form on $\mathcal{O}$, then $(\tilde{f})^*\omega$ extends to $\tilde{\omega} \in H^0(\hat{Z}, \Omega^2_{\hat{Z}})$. $\hat{Z}$ admits a $\mathbb{C}^*$-action induced from the one on W_0 and for this action, one has $\lambda^*\tilde{\omega} = \lambda\tilde{\omega}$ for all $\lambda \in \mathbb{C}^*$. By contracting $\tilde{\omega}$ with the vector field generating the $\mathbb{C}^*$-action, we obtain a 1-form θ' on $\tilde{Z}$ satisfying $\lambda^*\theta' = \lambda\theta'$, this gives an element θ in $H^0(Z, \Omega_Z(f^*L))$. Then θ is a contact form if and only if $\tilde{\omega}$ is a symplectic form. □

From now on, we let $\mathcal{O}$ be a nilpotent orbit such that $\mathbb{P}(\widetilde{\mathcal{O}})$ is singular.

Proposition 4.3. *Let $\bar{\pi}: X \to \mathbb{P}(\overline{\mathcal{O}})$ be a contact resolution and $\tilde{L} = \bar{\pi}^*(L)$ the contact line bundle on X. Then $(X, \tilde{L})$ is isomorphic to $(\mathbb{P}(T^*Y), \mathcal{O}_{\mathbb{P}(T^*Y)}(1))$ for some smooth projective variety Y.*

Proof. Note that $K_X \simeq \tilde{L}^{-n-1}$, where $n = (dim\mathcal{O})/2 - 1$. For a curve C in X, we have $K_X \cdot C = -(n+1)L \cdot \bar{\pi}_*[C]$, thus K_X is not nef. By [KPSW], X is either a Fano contact manifold or $(X, \tilde{L})$ is isomorphic to $(\mathbb{P}(T^*Y), \mathcal{O}_{\mathbb{P}(T^*Y)}(1))$ for some smooth projective variety Y.

The map $\bar{\pi}$ factorizes through the normalization, so we obtain a birational map $\nu : X \to \mathbb{P}(\widetilde{\mathcal{O}})$. By assumption, $\mathbb{P}(\widetilde{\mathcal{O}})$ is singular. Zariski's main theorem then implies that there exists a curve C contained in a fiber of ν. Now $K_X \cdot C = 0$, thus $-K_X$ is not ample, which shows that X is not Fano. □

Let us denote by $\pi_0 : \hat{X} \to W_0$ the symplectic resolution provided by Proposition 4.2. By lemma 4.1, $\hat{X}$ is isomorphic to $T^*Y \setminus Y$.

Lemma 4.4. *π_0 extends to a morphism $\pi : T^*Y \to \overline{\mathcal{O}}$.*

Proof. Note that π_0 lifts to a morphism $\hat{X} \to \widetilde{W_0}$, where $\widetilde{W_0}$ is the normalization of W_0, which gives a homomorphism $H^0(\widetilde{W_0}, \mathcal{O}_{\widetilde{W_0}}) \to H^0(\hat{X}, \mathcal{O}_{\hat{X}})$. Notice that $H^0(\widetilde{W_0}, \mathcal{O}_{\widetilde{W_0}}) = H^0(\widetilde{\mathcal{O}}, \mathcal{O}_{\widetilde{\mathcal{O}}})$ and $H^0(\hat{X}, \mathcal{O}_{\hat{X}}) = H^0(T^*Y, \mathcal{O}_{T^*Y})$. On the other hand, we have a natural morphism $T^*Y \to Spec(H^0(T^*Y, \mathcal{O}_{T^*Y}))$, which composed with the map $Spec(H^0(T^*Y, \mathcal{O}_{T^*Y})) \simeq Spec(H^0(\hat{X}, \mathcal{O}_{\hat{X}})) \to Spec(H^0(\widetilde{W_0}, \mathcal{O}_{\widetilde{W_0}})) \simeq Spec(H^0(\widetilde{\mathcal{O}}, \mathcal{O}_{\widetilde{\mathcal{O}}})) = \widetilde{\mathcal{O}} \to \overline{\mathcal{O}}$ gives π. □

Notice that π is a symplectic resolution of $\overline{\mathcal{O}}$, thus the main theorem in [Fu1] implies that π is isomorphic to the moment map of the G-action on $T^*(G/P)$ for some parabolic subgroup P in G. So we obtain

Theorem 4.5. *Let $\mathcal{O}$ be a nilpotent orbit in a semi-simple Lie algebra $\mathfrak{g}$ such that $\mathbb{P}(\widetilde{\mathcal{O}})$ is singular. Suppose that we have a contact resolution $\pi : Z \to \mathbb{P}(\overline{\mathcal{O}})$, then $Z \simeq \mathbb{P}(T^*(G/P))$ for some parabolic subgroup P in the adjoint group G of $\mathfrak{g}$ and the morphism π is the natural one.*

Now Proposition 2.2 implies the following

Corollary 4.6. *Suppose $\mathfrak{g}$ is simple. The projectivised nilpotent orbit closure $\mathbb{P}(\overline{\mathcal{O}})$ admits a contact resolution if and only if either*

(i) $\mathcal{O}$ is the minimal nilpotent orbit, or
(ii) $\mathfrak{g}$ is of type G_2 and $\mathcal{O}$ is of dimension 8, or
(iii) $\overline{\mathcal{O}}$ admits a symplectic resolution.

The classification of nilpotent orbits satisfying case (iii) has been done in [Fu1] and [Fu2]. For example, every projectivised nilpotent orbit closure in $\mathfrak{sl}_n$ admits a contact resolution, which is given by the projectivisation of cotangent bundles of some flag varieties.

Recall that the twistor space of a compact quaternion-Kähler manifold is a contact Fano manifold ([Sa]). One may wonder if a contact resolution of $\mathbb{P}(\overline{\mathcal{O}})$ could be the twistor space of a quaterion-Kähler manifold. Unfortunately, the answer to this is in general no, as shown by the following:

Proposition 4.7. *Let G be a simple complex Lie group with Lie algebra $\mathfrak{g}$ and P a parabolic sub-group of G. Then $\mathbb{P}(T^*(G/P))$ is a twistor space of a quaternion-Kähler manifold if and only if $G/P \simeq \mathbb{P}^n$ for some n.*

Proof. Recall that the image of the moment map $T^*(G/P) \to \mathfrak{g}$ is a nilpotent orbit closure $\overline{\mathcal{O}}$, which gives a generically finite morphism $\pi : \mathbb{P}(T^*(G/P)) \to \mathbb{P}(\overline{\mathcal{O}})$. There are two cases:

(i) there is a fiber of positive dimension, then as proved in Proposition 4.3, $\mathbb{P}(T^*(G/P))$ is not Fano.

(ii) every fiber of π is zero-dimensional, then π is a finite G-equivariant surjective morphism. If $\mathbb{P}(T^*(G/P))$ is Fano, then by Proposition 6.3 [Be], either $\pi = id$ and $\mathcal{O} = \mathcal{O}_{min}$ or π is one of the G-covering in the list of Brylinski-Kostant (see table 6.2 [Be]). In both cases, one has that $\mathbb{P}(T^*(G/P))$ is isomorphic to $\mathbb{P}(\mathcal{O}'_{min})$ for some minimal nilpotent orbit $\mathcal{O}'_{min} \subset \mathfrak{g}'$, which is possible only if G/P is isomorphic to $\mathbb{P}^n$ for some n.

Now suppose that $\mathbb{P}(T^*G/P)$ is a twistor space of a quaternion-Kähler manifold M. Then the scalar curvature of M would be positive, which implies ([Sa]) that $\mathbb{P}(T^*G/P)$ is Fano, so G/P is isomorphic to $\mathbb{P}^n$ for some n. □

As pointed out by Prof. A. Swann, this proposition follows also from [LeSa], where it is shown that a contact Fano variety with $b_2 \geq 2$ is isomorphic to $\mathbb{P}(T^*\mathbb{P}^n)$ for some n.

5 Birational geometry

Let $\mathfrak{g}$ be a simple complex Lie algebra and $\mathcal{O}$ a non-zero nilpotent orbit in $\mathfrak{g}$. We now try to understand the birational geometry between different contact resolutions of $\mathbb{P}(\overline{\mathcal{O}})$. We can assume that $\mathcal{O}$ is not the minimal nilpotent orbit, since $\mathbb{P}(\overline{\mathcal{O}}_{min})$ is smooth.

Suppose that $\overline{\mathcal{O}}$ admits a symplectic resolution, then by [Fu1], it is given by the natural map $\pi : X := T^*(G/P) \to \overline{\mathcal{O}}$ for some parabolic sub-group P in G. Let us denote by π_0 the restriction of π to $X_0 := T^*(G/P) \setminus (G/P)$, then π_0 is a symplectic resolution of $W_0 := \overline{\mathcal{O}} \setminus \{0\}$.

I'm indebted to M. Brion for the proof of the following proposition, which allows us to remove the restriction that $\mathfrak{g}$ is of classical type in an earlier version of this note.

Proposition 5.1. *We have $N_1(\pi_0) = N_1(\pi)$ and $N^1(\pi_0) = N^1(\pi)$.*

Proof. Consider the natural projections: $X_0 \xrightarrow{p_0} G/P \xleftarrow{p} X$, then $Pic(X_0) \otimes \mathbb{R}$ is identified with $Pic(G/P) \otimes \mathbb{R} = N^1(G/P)$ via p_0^*. Notice that for a complete curve C on X_0 and a divisor $D \in Pic(G/P)$, we have $C \cdot p_0^* D = (p_0)_*(C) \cdot D$. Thus we need to show that images of complete curves in X_0 under $(p_0)_*$ generate $H_2(G/P, \mathbb{R}) = N_1(G/P)$.

Let I be the set of simple roots of G which are not roots of the Levi subgroup of P, i.e. I is the set of marked roots in the marked Dynkin diagram of $\mathfrak{p} = \mathrm{Lie}(P)$. A basis of $H_2(G/P, \mathbb{R})$ is given by Schubert curves $C_\alpha := P_\alpha/B$, where $\alpha \in I$ and P_α is the corresponding minimal parabolic subgroup containing the Borel subgroup B. We need to lift every C_α to a curve in X_0. There are two cases:

(i) I consists of a single simple root α, then $b_2(G/P) = 1$. Since $\mathcal{O}$ is supposed to be non-minimal, and the 8-dimensional nilpotent orbit closure in G_2 has no symplectic resolution (Proposition 3.21 [Fu1]), by Proposition 2.2, we can assume that $\widetilde{\mathcal{O}} \setminus \{o\}$ is not smooth. By Zariski's main theorem, there exists a fiber of π_0 which has positive dimension. Take an irreducible curve C containing in this fiber, then $(p_0)_* C$ is non-zero in $H_2(G/P, \mathbb{R}) \simeq \mathbb{R}$, which generates (over $\mathbb{R}$) $N_1(G/P)$.

(ii) I contains at least two simple roots. To lift C_α, we take a simple root $\beta \in I$ different to α, then $\mathfrak{g}_\beta$ generates a G_α-submodule M of $\mathfrak{g}$ contained in $\mathfrak{n}$, where G_α is the simple subgroup of G associated with the simple root α and $\mathfrak{n}$ is the nilradical of $\mathfrak{p}$. Then in $T^*(G/P) \simeq G \times^P \mathfrak{n}$, there is the closed subvariety $P_\alpha \times^B M \simeq G_\alpha \times^{B_\alpha} M$ which is mapped to $G_\alpha M = M$ with fibers $G_\alpha/B_\alpha \simeq P_\alpha/B$, where $B_\alpha = B \cap G_\alpha$. Now any fiber of this map lifts C_α. $\square$

Let $\bar{\pi} : \mathbb{P}(X) \to \mathbb{P}(\overline{\mathcal{O}})$ be the induced map, which is a contact resolution.

The contact structure on $\mathbb{P}(X)$ is given by the line bundle $\tilde{L} = \mathcal{O}_{\bar{p}}(1)$, where $\bar{p} : \mathbb{P}(X) \to G/P$ is the natural map. We have $Pic(\mathbb{P}(X)) \simeq Pic(G/P) \oplus \mathbb{Z}[\tilde{L}]$. Notice that $\tilde{L} = \bar{\pi}^* L$, so for any $\bar{\pi}$-exceptional curve C, one has $C \cdot \tilde{L} = C \cdot \bar{\pi}^* L = 0$, so $\tilde{L}$ is zero in $N^1(\bar{\pi})$. This provides the identifications $N^1(\bar{\pi}) = N^1(\pi_0) = N^1(\pi)$ and $N_1(\bar{\pi}) = N_1(\pi_0) = N_1(\pi)$.

Recall that the cone $NE(\pi) = NE(G/P)$ is generated by Schubert curves in G/P over $\mathbb{R}^{\geq 0}$. As shown in the proof of Proposition 5.1, these Schubert curves are images of curves in the fibers of π_0, thus $NE(\pi_0) = NE(\pi)$. Since $\overline{NE(\pi_0)} = \overline{NE(\bar{\pi})}$, we obtain $NE(\bar{\pi}) = NE(\pi)$. By Kleiman's criterion, $\overline{Amp}(\pi_0) = \overline{Amp}(\pi) = \overline{Amp}(\bar{\pi})$. By [Na] Theorem 4.1 (ii), this is a simplicial polyhedral cone.

Let $g : X_0 \to \mathbb{P}(X)$ and $h : W_0 \to \mathbb{P}(\overline{\mathcal{O}})$ be the natural projections, then $\bar{\pi} g = h \pi_0$. Take a π_0-movable divisor $p_0^* D$, then $(\pi_0)_* p_0^* D = h^* \bar{\pi}_* \bar{p}^* D \neq 0$, which gives that $\bar{\pi}_* \bar{p}^* D \neq 0$. Notice that $\pi_0^* (\pi_0)_* p_0^* D = g^* \bar{\pi}^* \bar{\pi}_* \bar{p}^* D$ and $p_0^* D = g^* \bar{p}^* D$, so the cokernel of $\bar{\pi}^* \bar{\pi}_* \bar{p}^* D \to \bar{p}^* D$ has support of codimension ≥ 2. In conclusion $\bar{p}^* D$ is $\bar{\pi}$-movable and vice versa. So we obtain $\overline{Mov}(\pi_0) = \overline{Mov}(\pi) = \overline{Mov}(\bar{\pi})$.

To remember the parabolic subgroup P, from now on, we will write π_P instead of π (similarly for $\pi_0, \bar{\pi}$). For two parabolic subgroups Q, Q' in G, we write $Q \sim Q'$ (called *equivalent*) if $T^*(G/Q)$ and $T^*(G/Q')$ give both symplectic resolutions of a same nilpotent orbit closure. In [Na], Namikawa found a way to describe all parabolic subgroups which are equivalent to a given one. Furthermore the chamber structure of $\overline{Mov}(\pi_P)$ has been described in *loc. cit.* Theorem 4.1. By our precedent discussions $\overline{Mov}(\pi_0) = \overline{Mov}(\pi) = \overline{Mov}(\bar{\pi})$, thus we obtain the chamber structure of $\overline{Mov}(\bar{\pi})$, namely:

Theorem 5.2. *Let $\mathcal{O}$ be a non-minimal nilpotent orbit in a simple complex Lie algebra $\mathfrak{g}$ whose closure admits a symplectic resolution, say $T^*(G/P)$, where G is the adjoint group of $\mathfrak{g}$. Let $\bar{\pi}_P : \mathbb{P}(T^*(G/P)) \to \mathbb{P}(\overline{\mathcal{O}})$ be the associated contact resolution. Then $\overline{Mov}(\bar{\pi}_P) = \cup_{Q \sim P} \overline{Amp}(\bar{\pi}_Q)$.*

By Mori theory (see for example [Ma], Theorem 12-2-7), this implies that every minimal model of $\mathbb{P}(\overline{\mathcal{O}})$ is of the form $\mathbb{P}(T^*(G/Q))$ for some parabolic subgroup $Q \subset G$ such that $P \sim Q$. Now by Proposition 3.3, this gives another proof of Theorem 4.5 in the case where $\overline{\mathcal{O}}$ admits a symplectic resolution.

Similarly, as a by-product of our argument, we obtain the description of the movable cone of a symplectic resolution of W_0, which shows by Mori theory that every symplectic resolution of $\overline{\mathcal{O}} \setminus \{0\}$ is the restriction of a Springer map, thus

Corollary 5.3. *Let $\mathfrak{g}$ be a simple Lie algebra and $\mathcal{O} \subset \mathfrak{g}$ a nilpotent orbit. Suppose that $\overline{\mathcal{O}}$ admits a symplectic resolution, then every symplectic resolution of $\overline{\mathcal{O}} \setminus \{0\}$ can be extended to a symplectic resolution of $\overline{\mathcal{O}}$.*

Remark 5.1. The condition that $\overline{\mathcal{O}}$ admits a symplectic resolution cannot be removed, due to the following two examples:

(i). if $\mathfrak{g}$ is not of type A, then $\overline{\mathcal{O}}_{min}$ admits no symplectic resolution ([Fu1]), however $\overline{\mathcal{O}}_{min} - \{0\}$ is smooth, so trivially it admits a symplectic resolution;

(ii). if $\mathfrak{g}$ is of type G_2 and $\mathcal{O}$ is the 8-dimensional nilpotent orbit, then $W_0 := \overline{\mathcal{O}} - \{0\}$ is not smooth, and its normalization map $\mu : \widetilde{W_0} \to W_0$ is a symplectic resolution which does not extends to $\overline{\mathcal{O}}$, since $\mathcal{O}$ is not a Richardson nilpotent orbit (Prop. 3.21 [Fu1]). Here we used the result in [LeSm] and [Kr] which says that $\widetilde{W_0}$ is in fact the minimal nilpotent orbit in $\mathfrak{so}_7$, thus it is smooth and symplectic.

Are these two examples the only exceptions?

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